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# Distality Rank

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# Understanding Unstable NIP Theories

- Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.



# Understanding Unstable NIP Theories

- Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.
- It was motivated as an attempt to better understand unstable NIP theories by studying their stable and “purely unstable,” or *distal*, parts separately. This decomposition is particularly easy to see for **algebraically closed valued fields (ACVF)**:

**Stable Part:** Residue field

**Distal Part:** Value group

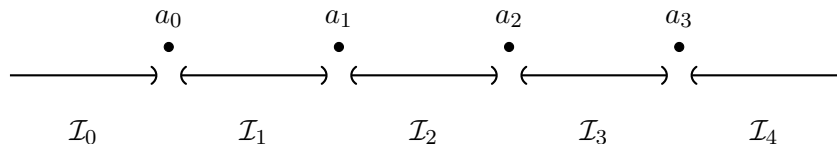


# Distality Rank for EM-Types



# 1-Distality in Pictures

A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$  is **1-distal** iff: for all  $A = (a_0, a_1, a_2, a_3)$ , if each **singleton** from  $A$  inserts indiscernibly ...

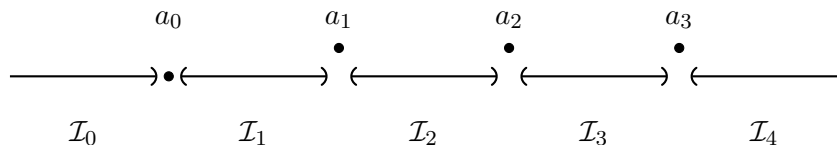


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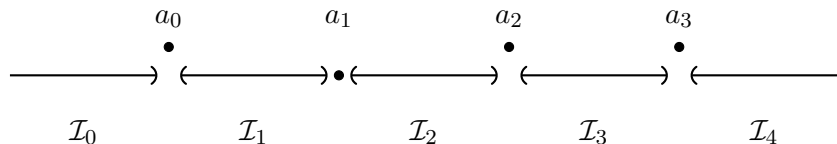


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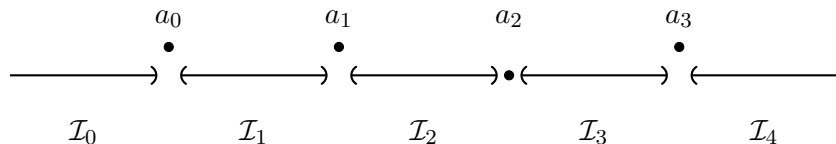


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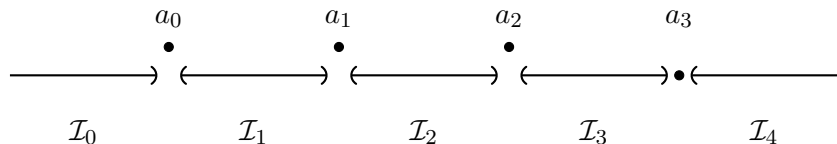


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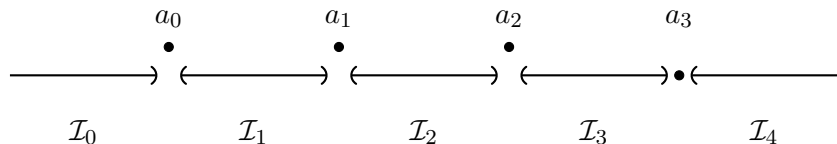


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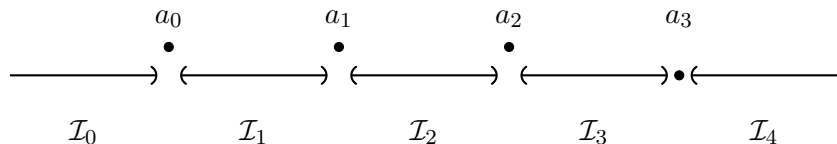


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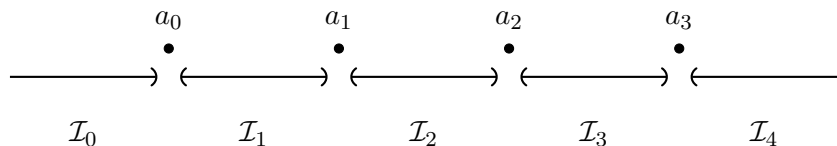


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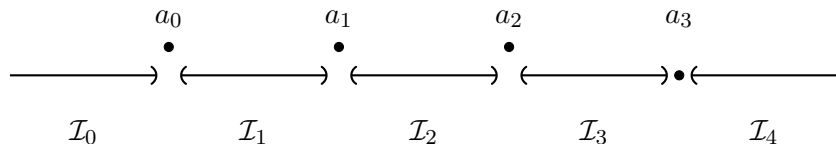


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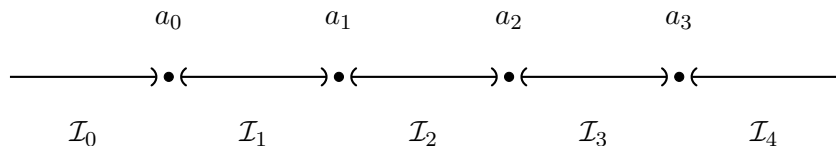


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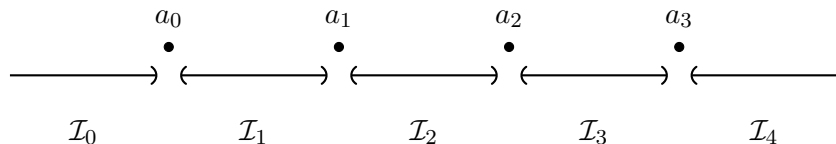


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## 2-Distality in Pictures

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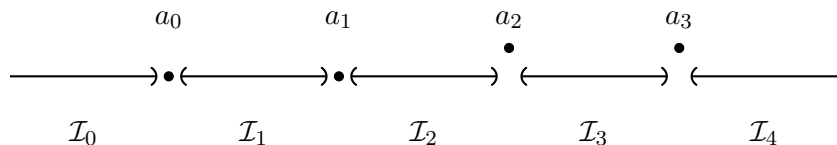


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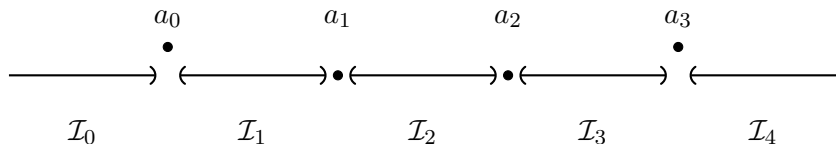


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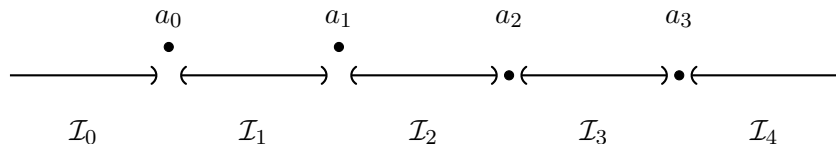


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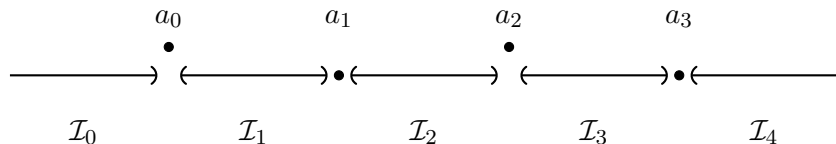
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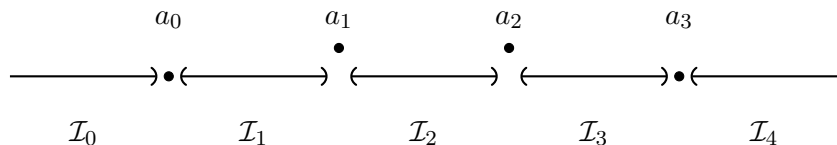


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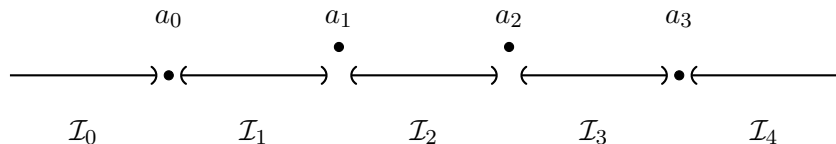


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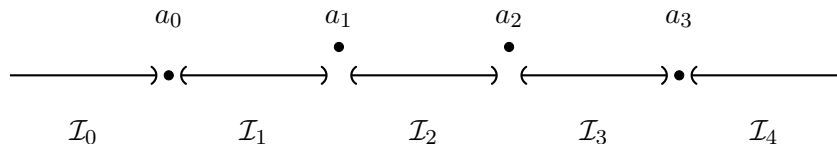


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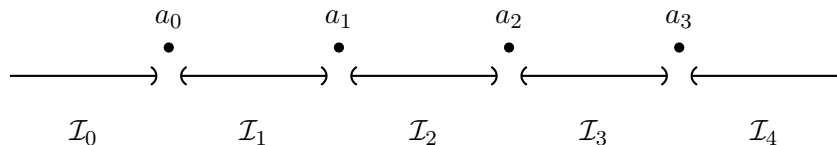


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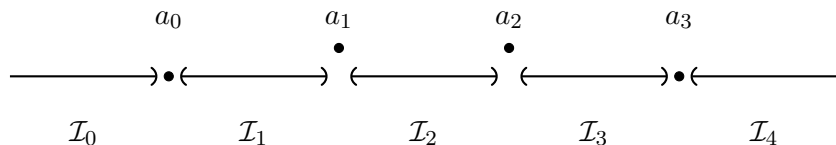


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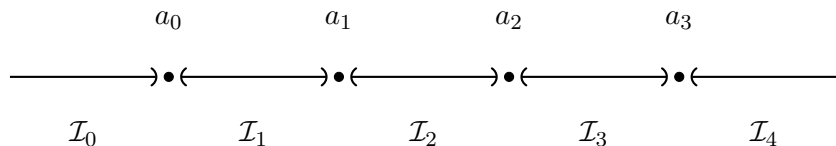


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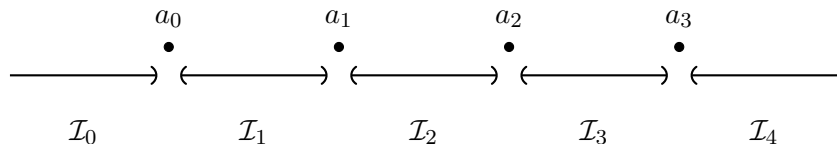


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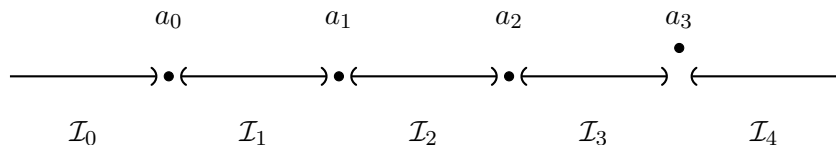


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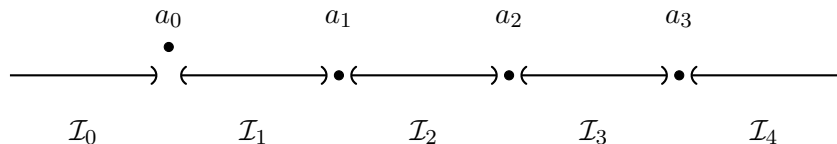


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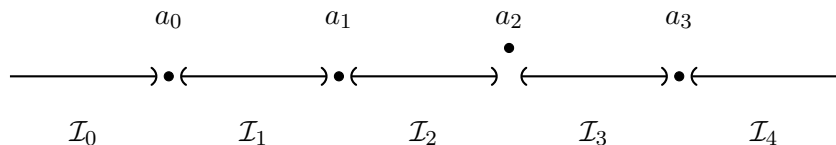


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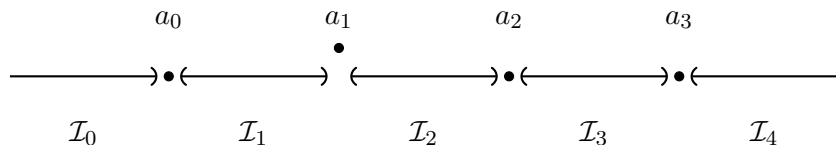


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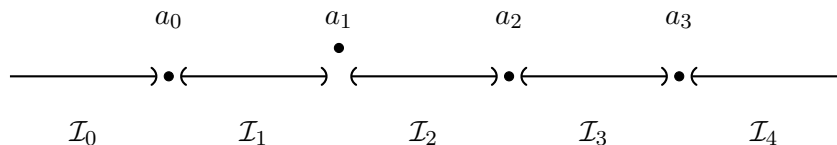


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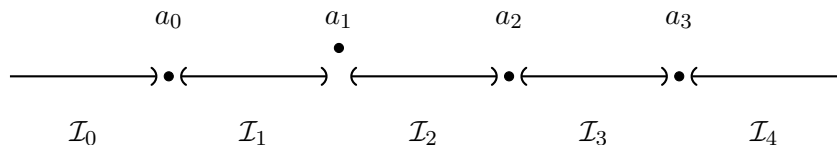


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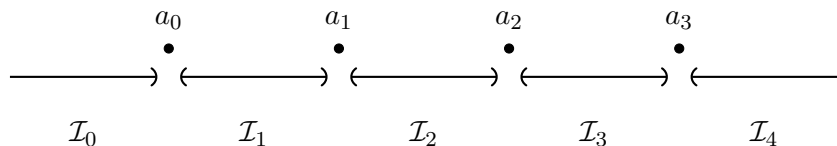


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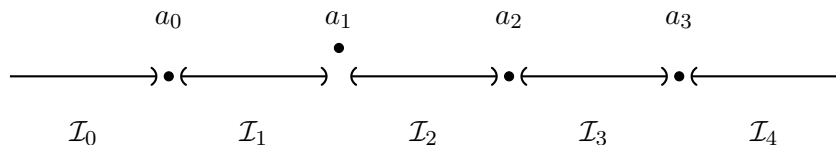


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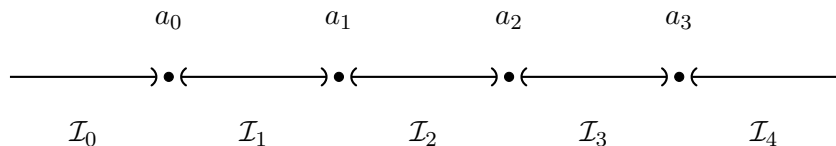


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# $m$ -Distality

Let  $n > m > 0$ .

## Definition

We say a Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_n$  is  *$m$ -distal* iff: for all sets  $A = (a_0, \dots, a_{n-1}) \subseteq U$ , if  $A$  does not insert indiscernibly into  $\mathcal{I}$ , then some  $m$ -element subset of  $A$  does not insert indiscernibly into  $\mathcal{I}$ .



# $m$ -Distality for EM-Types

Let  $n > m > 0$ .

## Definition

A complete EM-type  $\Gamma$  is  **$(n, m)$ -distal** iff: every Dedekind partition  $\mathcal{I}_0 + \cdots + \mathcal{I}_n \models^{\text{EM}} \Gamma$  is  $m$ -distal.

## Lemma

If  $\Gamma$  is  $(m + 1, m)$ -distal, then  $\Gamma$  is  $(n, m)$ -distal for all  $n > m$ .

**Proof:** Induction on  $n$ .



# $m$ -Distality for EM-types

## Definition

A complete EM-type  $\Gamma$  is  **$m$ -distal** iff: it is  $(m + 1, m)$ -distal.

## Theorem

Suppose  $T$  is NIP. A complete EM-type  $\Gamma$  is  $m$ -distal if and only if there is an  $m$ -distal Dedekind partition  $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\text{EM}} \Gamma$ .



# Distality Rank for EM-Types

**Observation:** If a complete EM-type  $\Gamma$  is  $m$ -distal, then it is also  $n$ -distal for all  $n > m$ .

## Definition

The *distality rank* of a complete EM-type  $\Gamma$ , written  $\text{DR}(\Gamma)$ , is the least  $m \geq 1$  such that  $\Gamma$  is  $m$ -distal. If no such finite  $m$  exists, we say the distality rank of  $\Gamma$  is  $\omega$ .



# Distality Rank for Theories



# Distality Rank for Theories

Let  $m > 0$ .

## Definition

A theory  $T$  is  *$m$ -distal* iff: in every model  $\mathcal{M} \models T$ , every indiscernible Dedekind partition is  $m$ -distal.

In the existing literature, a theory is called distal if and only if it is 1-distal.

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## Finding Examples

### Proposition

Suppose  $\mathcal{L}$  is a language where all function symbols are unary and all relation symbols have arity at most  $m \geq 2$ . If  $T$  is an  $\mathcal{L}$ -theory with quantifier elimination, then  $\text{DR}(T) \leq m$ .



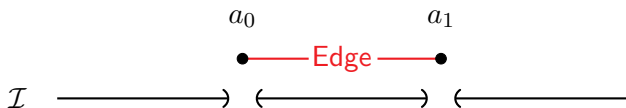
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This proposition helps us find examples by putting an upper bound on distality rank:

- The theory of the random graph has distality rank 2.



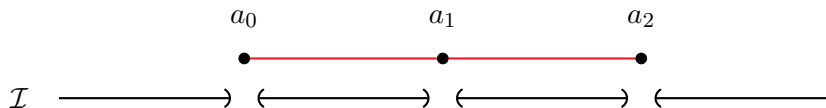
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**This generalizes, so...**

- The theory of the random  $m$ -hypergraph has distality rank  $m$ .



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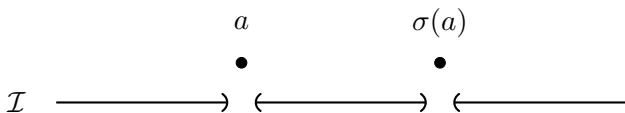
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- The theory of the random  $m$ -hypergraph has distality rank  $m$ .
- The theories of  $(\mathbb{N}, \sigma, 0)$  and  $(\mathbb{Z}, \sigma)$ , where  $\sigma : x \mapsto x + 1$ , have distality rank 2.



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We can not apply the proposition to groups...



For example, if  $T$  is the complete theory of a **strongly minimal group**, then  $\text{DR}(T) = \omega$ :

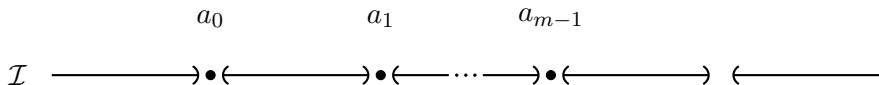


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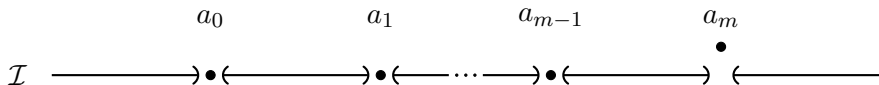
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Let  $a_m = a_0 + \cdots + a_{m-1}$ , and let  $A = (a_0, \dots, a_m)$ .



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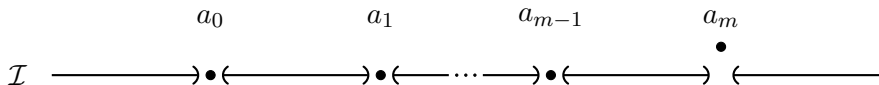


For example, if  $T$  is the complete theory of a **strongly minimal group**, then  $\text{DR}(T) = \omega$ :

Let  $\mathcal{I}a_0 \cdots a_{m-1}$  be an algebraically independent set.

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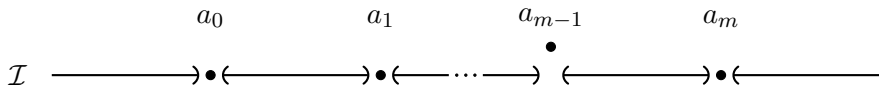


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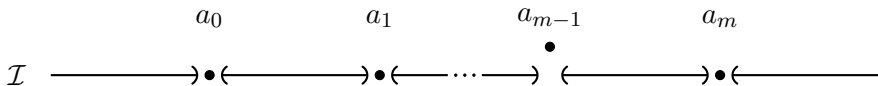


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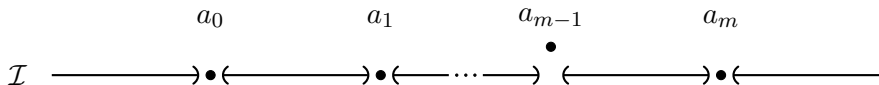


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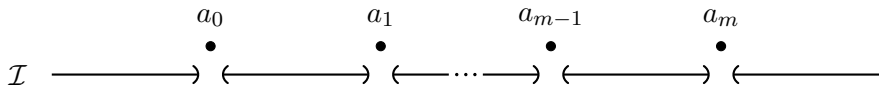
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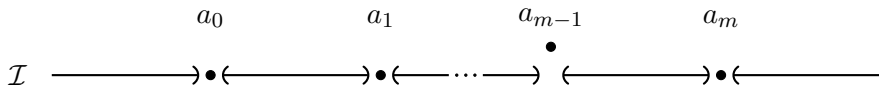


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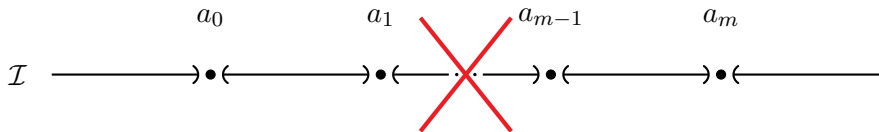
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Reset

# Properties of Distality Rank



# Base Change

Adding named parameters does not increase distality rank.

## Proposition

If  $T$  is a complete theory and  $B \subseteq U$  is a small set of parameters, then  $\text{DR}(T_B) \leq \text{DR}(T)$ .



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If  $T$  is NIP, adding named parameters does not change distality rank.

## Base Change Theorem

If  $T$  is NIP and  $B \subseteq U$  is a small set of parameters, then  $\text{DR}(T_B) = \text{DR}(T)$ .



# Type Determinacy

Let  $n > m > 0$ .

## Definition

Given  $p \in S_A(x_0, \dots, x_{n-1})$ , we say that the  $n$ -type  $p$  is ***m-determined*** iff: it is completely determined by the  $m$ -types

$$\{q \in S_A(x_{i_0}, \dots, x_{i_{m-1}}) : q \subseteq p \text{ and } i_0 < \dots < i_{m-1} < n\}$$

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## Theorem

If  $T$  is  $m$ -distal, then for any  $n$  global invariant types

$$p_0(x_0), \dots, p_{n-1}(x_{n-1})$$

which commute pairwise, their product  $p_0 \otimes \dots \otimes p_{n-1}$  is  $m$ -determined.

Furthermore, if  $T$  is NIP, the converse holds as well.



# Type Determinacy for Stable Theories

In the stable context, we can characterize  $m$ -distality using the determinacy of powers rather than mixed-factor products.

## Proposition

Suppose  $T$  is stable. Given  $m > 0$ , the following are equivalent:

- $T$  is  $m$ -distal.
- For every global invariant type  $p \in S_U$  and every  $n > m$ , the product  $p^n$  is  $m$ -determined.
- For every global invariant type  $p \in S_U$  and every  $n > m$ , the product  $p^n$  is  $(n - 1)$ -determined.



# The Stable Context: $k$ -Triviality



# $k$ -Triviality

Suppose  $T$  is stable.

## Definition

We say  $T$  is  **$k$ -trivial** iff: for all  $a \in U$  and  $B, D \subseteq U$ ,

$$a \underset{D}{\not\ll} B \implies a \underset{D}{\not\ll} \bar{b}$$

for some  $\bar{b} \in B^k$ . Moreover, we say  $T$  is **trivial** iff: it is 1-trivial.

It follows that  $T$  is  $k$ -trivial if and only if there are no  $(k + 2)$ -cycles (i.e., minimal dependent sets of size  $k + 2$ ).



## Relationship between $m$ -Distality and $k$ -Triviality

Suppose  $T$  is stable.

### Theorem

$T$  is  $k$ -trivial if and only if it is  $(k + 1)$ -distal.

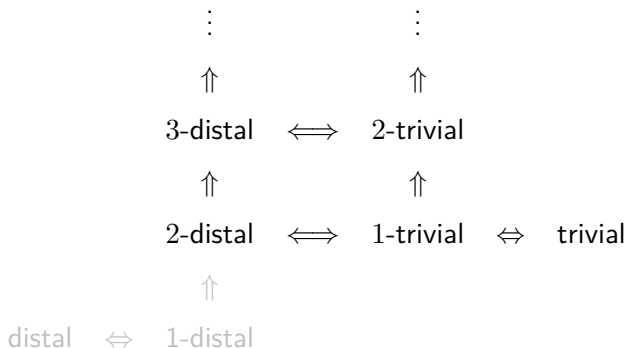


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In *Some Trivial Considerations*, John B. Goode (aka Poizat) shows that for any finite  $k$ , a superstable  $k$ -trivial theory is trivial.

### Corollary

Suppose  $T$  is **superstable**.

- If  $T$  is trivial, then  $\text{DR}(T) = 2$ .
- Otherwise,  $\text{DR}(T) = \omega$ .



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Whether or not this dichotomy persists for stable theories is equivalent to a long-standing open question: (Goode 1991) Can a stable theory be  $k$ -trivial for some  $k > 1$  but not trivial?

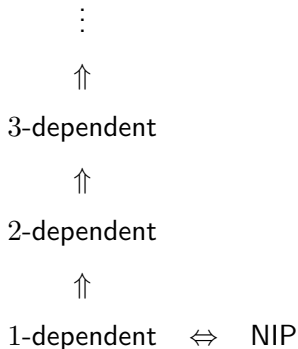


# The General Context: $n$ -Dependence



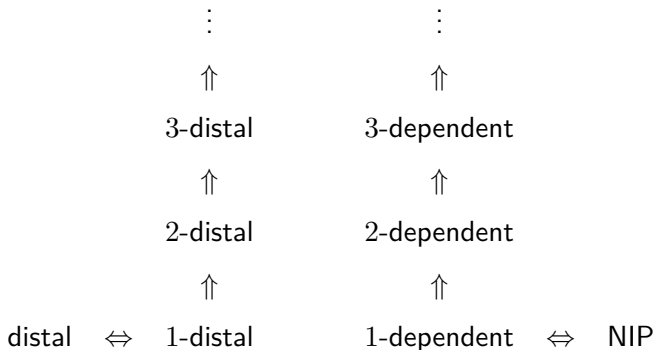
# Relationship between $m$ -Distality and $n$ -Dependence

Shelah introduced  *$n$ -dependence* as a property of first-order theories which generalizes NIP:



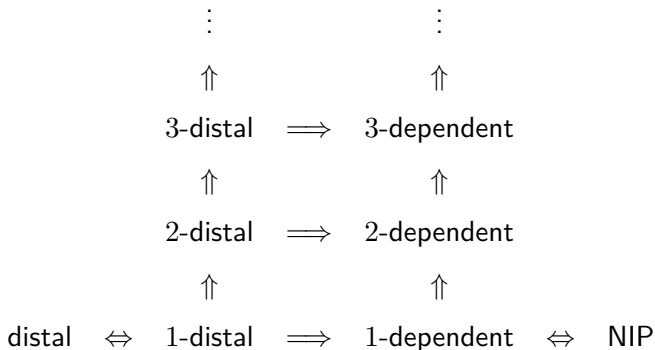
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Artem Chernikov observed that  $m$ -distality implies  $m$ -dependence.

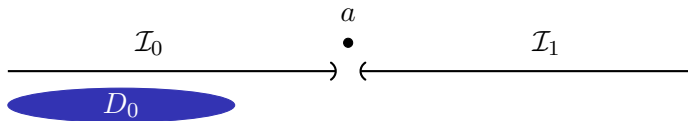


# Strong Distality Rank for EM-Types



A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly 1-distal** iff:

for all  $a \in U$  and all small bases  $D_0 \subseteq U$ , if  $\mathcal{I}$  is indiscernible over  $D_0 \dots$

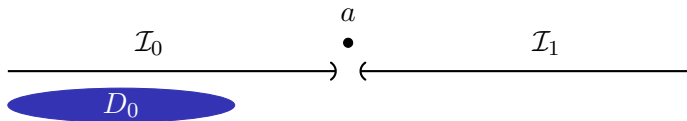


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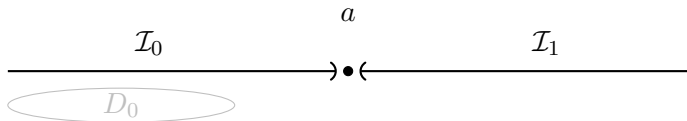


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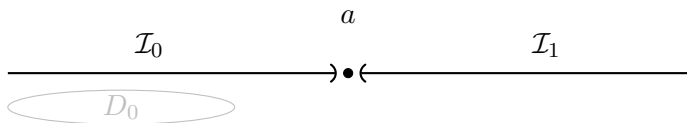


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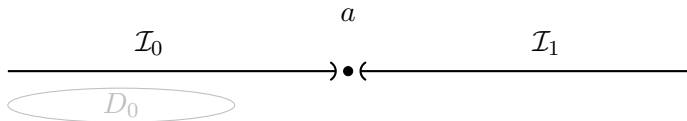


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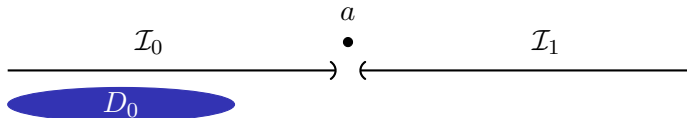


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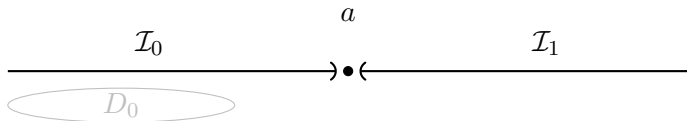


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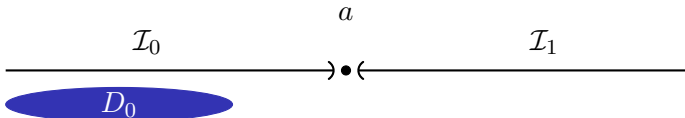


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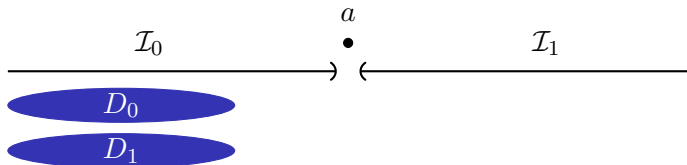


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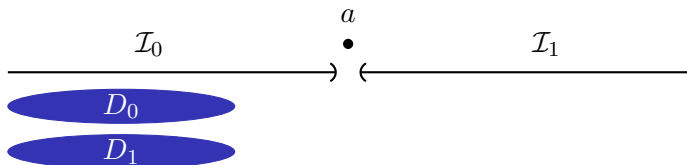


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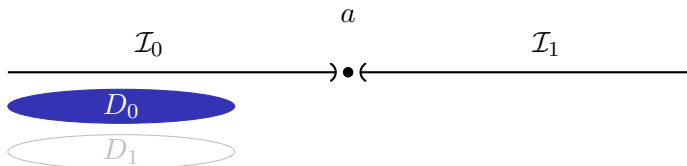


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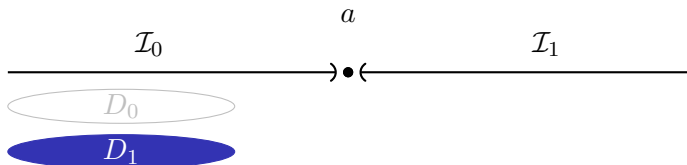


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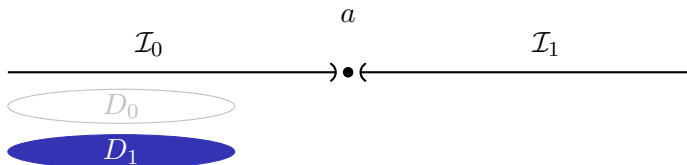


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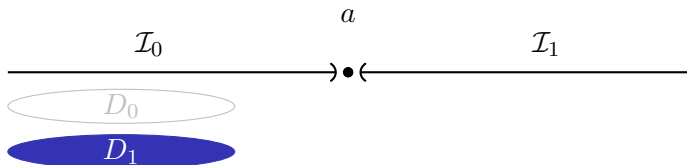


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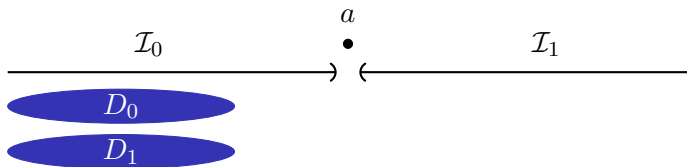


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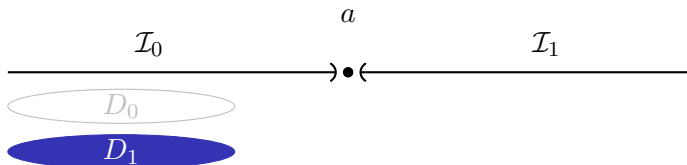


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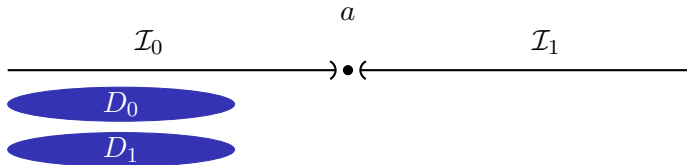


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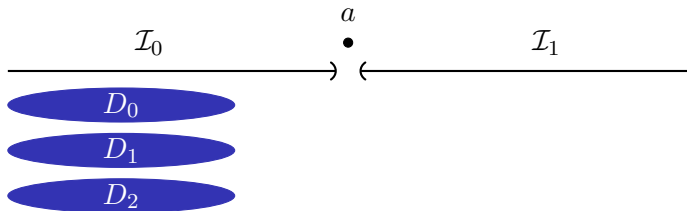


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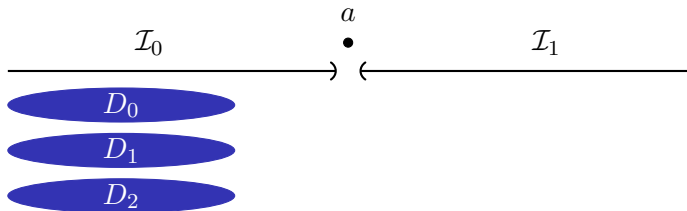


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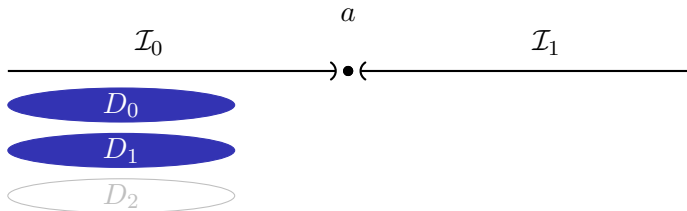


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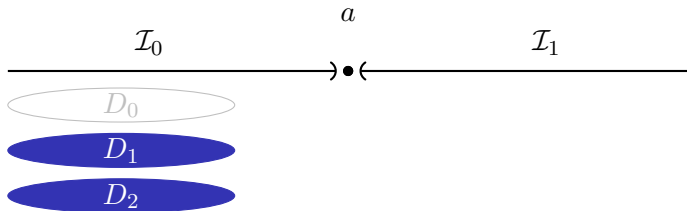


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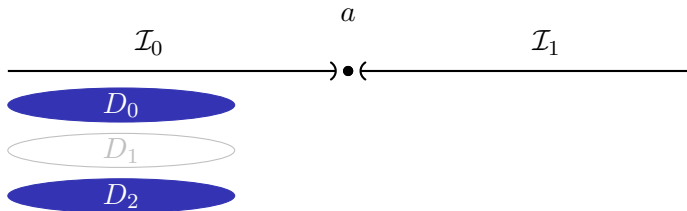


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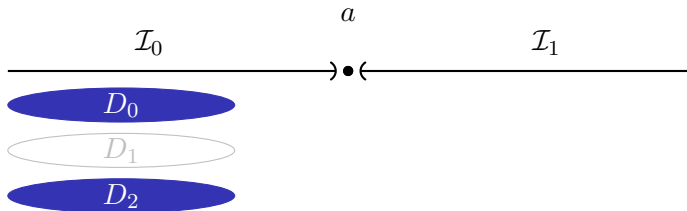


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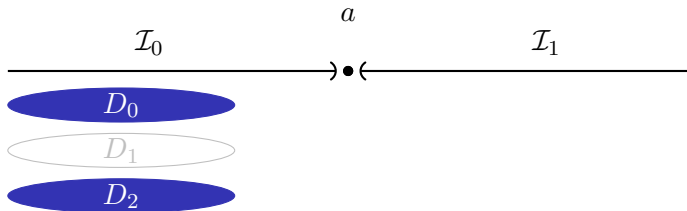


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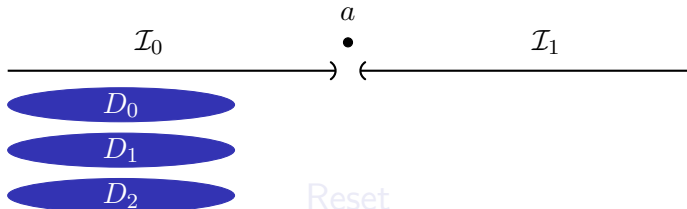


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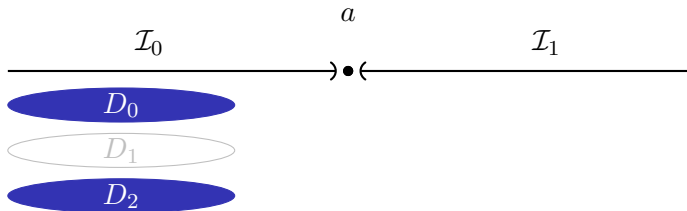
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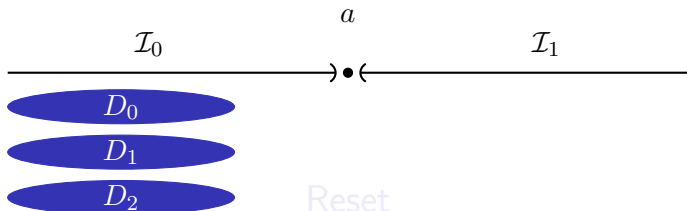
Reset

A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly 3-distal** iff:

for all  $a \in U$  and all small bases  $D_0, D_1, D_2 \subseteq U$ , if  $\mathcal{I}$  is indiscernible over  $\bar{D}$  and  $a$  inserts over each pair  $D_0D_1$ ,  $D_1D_2$ , and  $D_0D_2$ ,



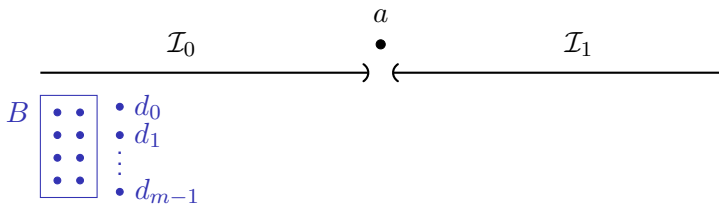
then  $a$  inserts indiscernibly over  $\bar{D}$ .



Reset

A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly  $m$ -distal** iff:

for all  $a \in U$ , all finite bases  $B \subseteq U$ , and all  $\bar{d} \in [U]^m$ , if  $\mathcal{I}$  is indiscernible over  $B\bar{d}$  ...



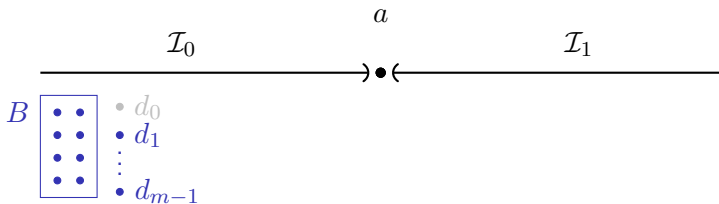
Reset





A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly  $m$ -distal** iff:

for all  $a \in U$ , all finite bases  $B \subseteq U$ , and all  $\bar{d} \in [U]^m$ , if  $\mathcal{I}$  is indiscernible over  $B\bar{d}$  and  $a$  inserts indiscernibly over each  $B\bar{d} \setminus d_i \dots$

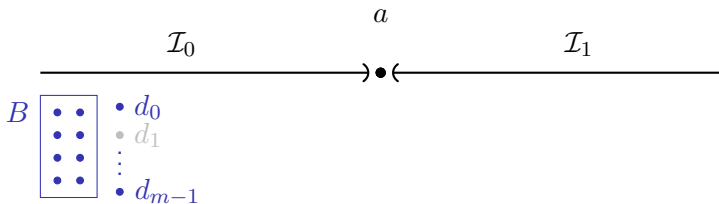


Reset



A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly  $m$ -distal** iff:

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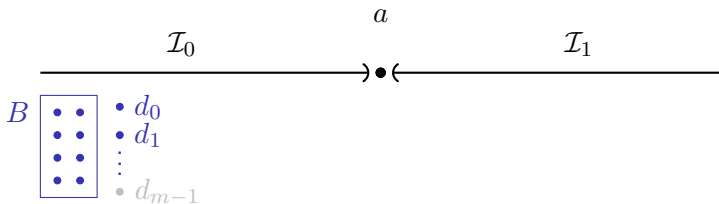


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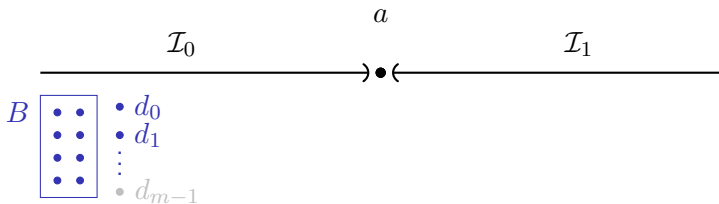


Reset



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then  $a$  inserts indiscernibly over  $B\bar{d}$  ...

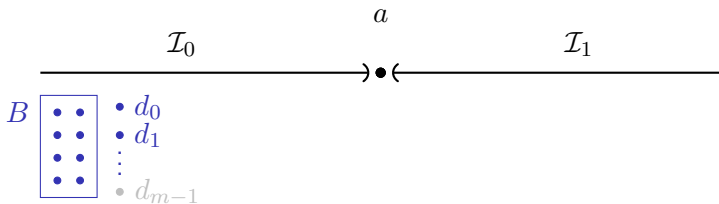


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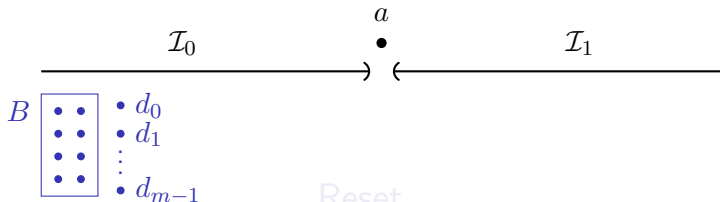


A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly  $m$ -distal** iff:

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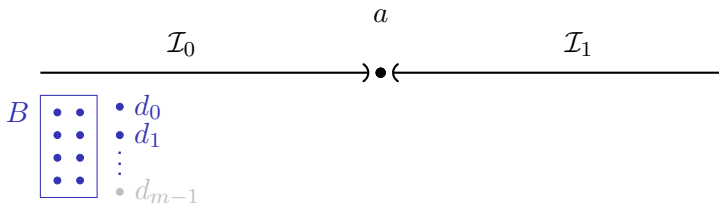
then  $a$  inserts indiscernibly over  $B\bar{d} \dots$



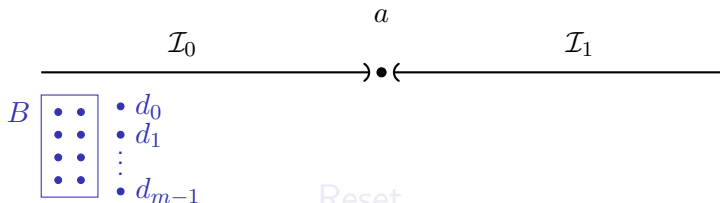
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A Dedekind partition  $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$  is **strongly  $m$ -distal** iff:

for all  $a \in U$ , all finite bases  $B \subseteq U$ , and all  $\bar{d} \in [U]^m$ , if  $\mathcal{I}$  is indiscernible over  $B\bar{d}$  and  $a$  inserts indiscernibly over each  $B\bar{d} \setminus d_i$ ,



then  $a$  inserts indiscernibly over  $B\bar{d}$ .



Reset

# Strong Distality Rank for EM-Types

Let  $m > 0$ .

## Definition

A complete EM-type  $\Gamma$  is **strongly  $m$ -distal** iff: every/some Dedekind partition  $\mathcal{I}_0 + \mathcal{I}_1 \models^{\text{EM}} \Gamma$  is strongly  $m$ -distal.

## Definition

The **strong distality rank** of a complete EM-type  $\Gamma$ , written  $\text{SDR}(\Gamma)$ , is the least  $m \geq 1$  such that  $\Gamma$  is strongly  $m$ -distal. If no such finite  $m$  exists, we say the strong distality rank of  $\Gamma$  is  $\omega$ .

## Proposition

$$\text{DR}(\Gamma) \leq \text{SDR}(\Gamma)$$



## Example: $\text{DR}(\Gamma) < \text{SDR}(\Gamma)$

Let  $\mathcal{L} = \{E, <, P_{\text{red}}, P_{\text{blue}}\}$ , and let  $T$  be the theory of the *ordered random bipartite graph*. If  $\Gamma$  is the EM-type of an increasing sequence of singletons in  $P_{\text{red}}$ , the  $\text{DR}(\Gamma) = 1$ .

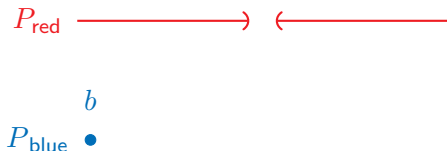


## Example: $\text{DR}(\Gamma) < \text{SDR}(\Gamma)$

Let  $\mathcal{L} = \{E, <, P_{\text{red}}, P_{\text{blue}}\}$ , and let  $T$  be the theory of the *ordered random bipartite graph*. If  $\Gamma$  is the EM-type of an increasing sequence of singletons in  $P_{\text{red}}$ , the  $\text{DR}(\Gamma) = 1$ .



But  $\text{SDR}(\Gamma) > 1$ .

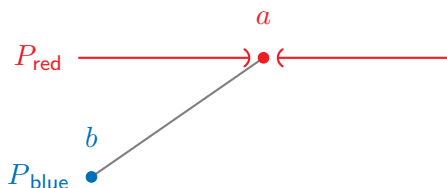


## Example: $\text{DR}(\Gamma) < \text{SDR}(\Gamma)$

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But  $\text{SDR}(\Gamma) > 1$ .



# Strong Distality Rank for Theories



# Strong Distality Rank for Theories

Let  $m > 0$ .

## Definition

A theory  $T$  is **strongly  $m$ -distal** iff: in every model  $\mathcal{M} \models T$ , every indiscernible Dedekind partition is strongly  $m$ -distal.

## Definition

The **strong distality rank** of a theory  $T$ , written  $\text{SDR}(T)$ , is the least  $m \geq 1$  such that  $T$  is strongly  $m$ -distal. If no such finite  $m$  exists, we say the strong distality rank of  $T$  is  $\omega$ .



# Strong Distality Rank for Invariant Types

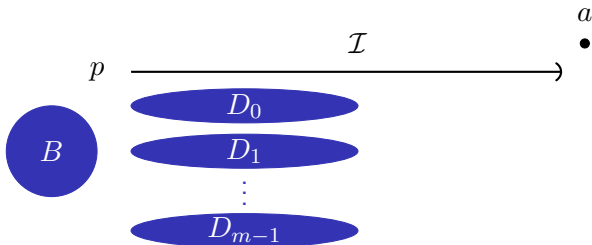
Fix a global type  $p \in S_U(x)$  which is invariant over some small base  $B \subseteq U$ .



$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,

if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}} \dots$

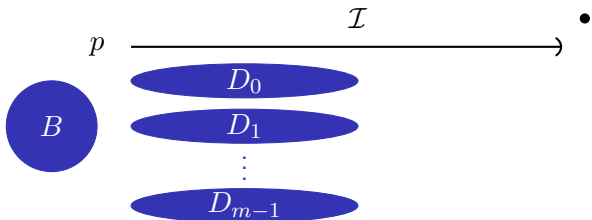


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$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
 if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}}$  and  $\mathcal{I} + a$  is indiscernible over each  $B\bar{D} \setminus D_i \dots$

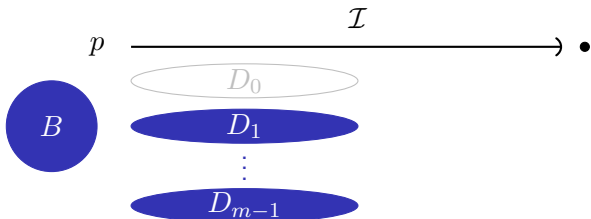


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$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
 if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}}$  and  $\mathcal{I} + a$  is indiscernible over each  $B\bar{D} \setminus D_i \dots$   
 $a$

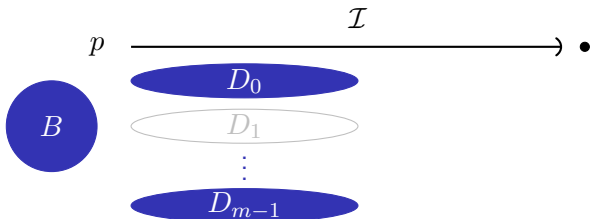


Reset



$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
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 $a$

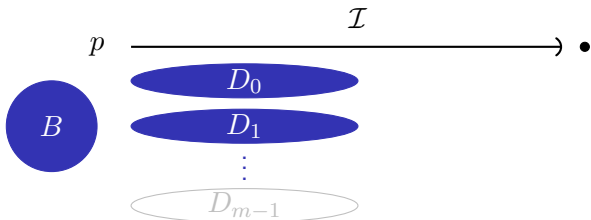


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$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
 if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}}$  and  $\mathcal{I} + a$  is indiscernible over each  $B\bar{D} \setminus D_i \dots$   
 $a$

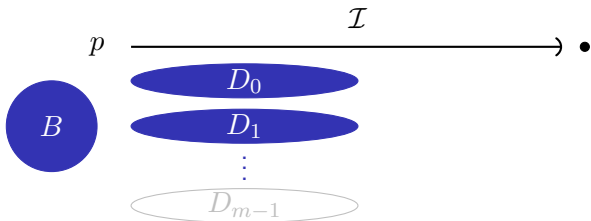


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$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
 if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}}$  and  $\mathcal{I} + a$  is indiscernible over each  $B\bar{D} \setminus D_i$ ,



then  $\mathcal{I} + a$  is indiscernible over  $B\bar{D}$  ...

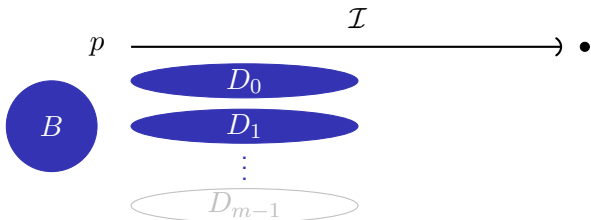


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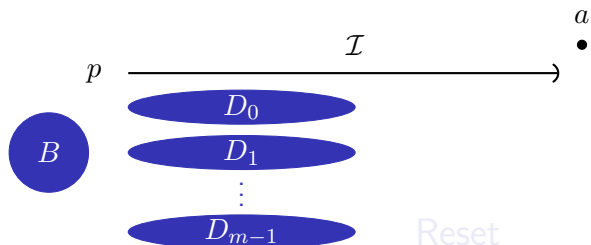


$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
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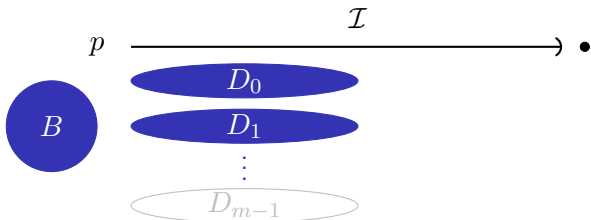
then  $\mathcal{I} + a$  is indiscernible over  $B\bar{D}$  ...



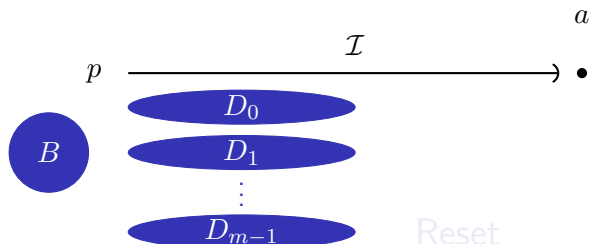
Reset

$p$  is **strongly  $m$ -distal over  $B$**  iff:

for all  $a \in U$  and all small disjoint bases  $D_0, \dots, D_{m-1} \subseteq U$ ,  
 if  $\mathcal{I} \models p^\omega \downarrow_{B\bar{D}}$  and  $\mathcal{I} + a$  is indiscernible over each  $B\bar{D} \setminus D_i$ ,



then  $\mathcal{I} + a$  is indiscernible over  $B\bar{D}$ .



Reset

# Strong Distality Rank for Invariant Types

## Definition

We say  $p$  is **strongly  $m$ -distal** iff: it is strongly  $m$ -distal over all of its invariance bases.

## Definition

The **strong distality rank** of  $p$ , written  $\text{SDR}(p)$ , is the least  $m \geq 1$  such that  $p$  is strongly  $m$ -distal. If no such finite  $m$  exists, we say the strong distality rank of  $p$  is  $\omega$ .

## Proposition

The theory  $T$  is strongly  $m$ -distal if and only if all its global invariant types are strongly  $m$ -distal.



# Strong Distality Rank and Type Determinacy

Let  $n \geq m > 0$ .

## Proposition

Given a global invariant type  $q \in S_U(y_0, \dots, y_{n-1})$ , if  $p$  is strongly  $m$ -distal over  $B$  and  $p \otimes q = q \otimes p$ , then their product is determined by  $q$  and restrictions of the form

$$(p \otimes q) \upharpoonright_{xy_{\sigma(0)} \dots y_{\sigma(m-2)}}$$

where  $\sigma : (m-1) \rightarrow n$ .



# Strong Distality Rank for Formulas



Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

## Definition

Given  $a, a' \in U^x$  and  $m \leq n$ , let

$$a \stackrel{m-1}{\equiv}_B a'$$

denote that

$$a_{\sigma(0)} \cdots a_{\sigma(m-2)} \stackrel{\equiv}{B} a'_{\sigma(0)} \cdots a'_{\sigma(m-2)}$$

for each  $\sigma : m-1 \rightarrow n$ .



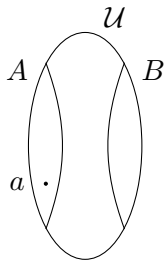
Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

### Definition (A)

The formula  $\phi$  is **strongly  $m$ -distal** iff: for every  $a \in U^x$  and  $B \subseteq U$ , if we let

$$A = \left\{ a' \in U^x : a' \stackrel{m-1}{\equiv}_B a \right\},$$

then there is a formula  $\theta(x, z) \in L(x, *)$  such that for all finite subsets  $A_0 \subseteq A$  and  $B_0 \subseteq B$ , we can find  $b \in B$  with  $a \in \theta(A, b)$  such that for each  $a' \in \theta(A_0, b)$ , we have  $\phi(a', B_0) = \phi(a, B_0)$ .



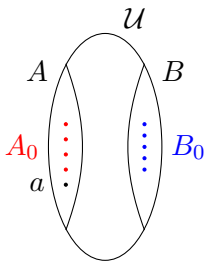
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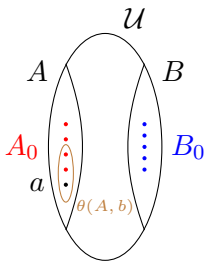
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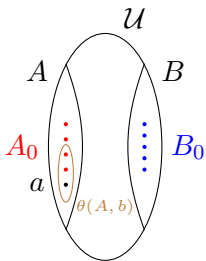
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- This property only depends on  $T$  and  $\phi$  (and how its variables are partitioned).
- It does *not* depend on our choice of monster model. Indeed, we may replace  $\mathcal{U}$  with any  $|T|^+$ -universal model.

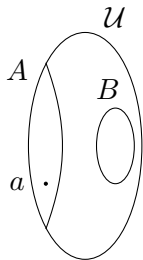
Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

## Definition (B)

The formula  $\phi$  is **strongly  $m$ -distal** iff: for every  $a \in U^x$  and **small**  $B \subseteq U$ , if we let

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then there is a formula  $\theta(x, z) \in L(x, *)$  such that for all finite subsets  $A_0 \subseteq A$  and  $B_0 \subseteq B$ , we can find  $b \in B$  with  $a \in \theta(A, b)$  such that for each  $a' \in \theta(A, b)$ , we have  $\phi(a', B_0) = \phi(a, B_0)$ .



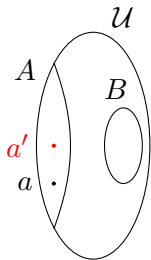
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- For each  $a' \in A$ , choose  $\theta_{a'}$  as above.

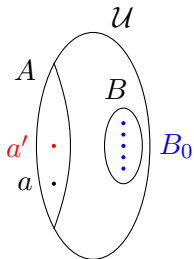
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then there is a formula  $\theta(x, z) \in L(x, *)$  such that for all finite subsets  $A_0 \subseteq A$  and  $B_0 \subseteq B$ , we can find  $b \in B$  with  $a \in \theta(A, b)$  such that for each  $a' \in \theta(A, b)$ , we have  $\phi(a', B_0) = \phi(a, B_0)$ .



- For each  $a' \in A$ , choose  $\theta_{a'}$  as above.
- Fix finite  $B_0 \subseteq B$ .

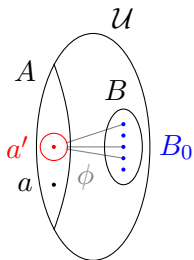
Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

## Definition (B)

The formula  $\phi$  is **strongly  $m$ -distal** iff: for every  $a \in U^x$  and **small**  $B \subseteq U$ , if we let

$$A = \left\{ a' \in U^x : a' \stackrel{m-1}{\equiv}_B a \right\},$$

then there is a formula  $\theta(x, z) \in L(x, *)$  such that for all finite subsets  $A_0 \subseteq A$  and  $B_0 \subseteq B$ , we can find  $b \in B$  with  $a \in \theta(A, b)$  such that for each  $a' \in \theta(A, b)$ , we have  $\phi(a', B_0) = \phi(a, B_0)$ .



- For each  $a' \in A$ , choose  $\theta_{a'}$  as above.
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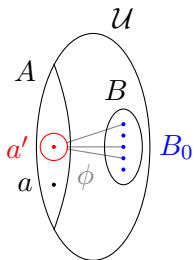
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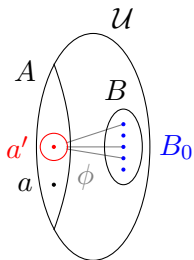
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- We have covered  $A$  with  $\phi$ -homogeneous cells of the form  $\theta_{a'}(x, b_{a'})$ .
- Since  $A/\equiv_B$  is a compact subspace of  $S_B(x)$ , there is a finite subcover.

Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

### Definition (C)

The formula  $\phi$  is **strongly  $m$ -distal** iff: for every small  $\mathcal{M} \models T$ , parameter  $a \in M^x$ , and set  $B \subseteq M$ , if we let

$$A = \left\{ a' \in M^x : a' \stackrel{m-1}{\equiv}_B a \right\},$$

then there is a formula  $\theta(x, z) \in L(x, *)$  such that for every  $|T_{AB}|^+$ -saturated elementary extension  $(M', A', B') \succ (M, A, B)$ , we can find a parameter  $b' \in B'$  with  $a \in \theta(A, b')$  such that for all  $a' \in \theta(A, b')$ , we have  $\phi(a', B) = \phi(a, B)$ .

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### Proposition

Definitions (A), (B), and (C) are equivalent.

Fix a formula  $\phi(x, y) \in L$  and a partition  $x_0, \dots, x_{n-1}$  of the variable  $x$ .

## Definition

The **strong distality rank** of  $\phi$ , written  $\text{SDR}(\phi)$ , is the least  $m \geq 1$  such that  $\phi$  is strongly  $m$ -distal. If no such finite  $m$  exists, we say the strong distality rank of  $\phi$  is  $\omega$ .

## Theorem

Given  $m > 0$ , the following are equivalent:

- The theory  $T$  is strongly  $m$ -distal.
- Every formula  $\psi(x, y) \in L$  is strongly  $m$ -distal for every finite partition  $x_0, \dots, x_{n-1}$  of  $x$  where  $n \geq m$ .
- Every formula  $\psi(x_0, \dots, x_{m-1}, y) \in L(1, \dots, 1, *)$  is strongly  $m$ -distal.

# Thank You!

A link to the paper and a longer version of the slides can be found at my website...

<https://homepages.math.uic.edu/~roland/>



## Adding Linear Orders to Stable Theories

Let  $L = \{\sim\}$ , and let  $T$  assert that  $\sim$  is an equivalence relation with infinitely many equivalence classes, each of which is infinite. It follows that  $T$  is **stable** and  $\text{DR}(T) = 2$ .

Let  $L' = \{\sim, <\}$ , and let  $\mathbb{K}$  be the set of all finite  $L'$ -structures where  $\sim$  is an equivalence relation and  $<$  is a linear order. If  $T_1$  is the Fraïssé limit of  $\mathbb{K}$ , then  $T_1$  is **unstable** and  $\text{DR}(T_1) = 2$ .

Let  $T_2 = \text{Th}(\mathbb{Q} \times \mathbb{Q}, \sim, <)$  where

$$(a_0, a_1) \sim (b_0, b_1) \iff a_0 = b_0$$

and

$$(a_0, a_1) < (b_0, b_1) \iff a_0 < b_0 \text{ or } [a_0 = b_0 \text{ and } a_1 < b_1].$$

It follows that  $T_2$  is **unstable** and  $\text{DR}(T_2) = 1$ .



# Appendix



## Proof:

( $\Leftarrow$ ) Suppose  $\Gamma$  is not  $m$ -distal. Let  $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\text{EM}} \Gamma$  be an  $(m+1)$ -skeleton. We will show that the skeleton is not  $m$ -distal.

Since  $\Gamma$  is not  $m$ -distal, there exist  $\mathcal{J} \models^{\text{EM}} \Gamma$ , a Dedekind partition  $\mathcal{J} = \mathcal{J}_0 + \cdots + \mathcal{J}_{m+1}$ , and a sequence  $A = (a_0, \dots, a_m) \in U$  such that all  $m$ -sized subsets insert but  $A$  does not. Let  $\phi \in \Gamma$  and  $\bar{b}_i \in \mathcal{J}_i$  such that

$$\mathcal{U} \not\models \phi(\bar{b}_0, a_0, \dots, \bar{b}_m, a_m, \bar{b}_{m+1}).$$

Construct  $\sigma : \mathcal{I} \rightarrow \mathcal{J}$  an order-preserving map such that

$$\bar{b}_i \subseteq \sigma(\mathcal{I}_i) \subseteq \mathcal{J}_i.$$

We can extend  $\sigma$  to an automorphism of  $\mathcal{U}$ . Let

$$A' = (\sigma^{-1}(a_0), \dots, \sigma^{-1}(a_m)).$$

Now any  $m$ -sized subset of  $A'$  inserts into  $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1}$ , but  $A'$  does not. ■



Return



## Proof of Lemma:

We will only handle the case where  $\mathcal{I}$  is dense.

Assume no such  $A'$  exists.

By compactness, there are  $\phi \in \text{tp}_{\mathcal{I}}(a_0, \dots, a_m)$  and  $\psi_\sigma \in \text{limtp}_D(\mathfrak{c}_{\sigma(0)}^-, \dots, \mathfrak{c}_{\sigma(m-1)}^-)$  for each  $\sigma : m \rightarrow m+1$  increasing such that

$$\phi(x_0, \dots, x_m) \vdash \bigvee_{\sigma} \neg \psi_\sigma(x_{\sigma(0)}, \dots, x_{\sigma(m-1)}). \quad (*)$$

Let  $B \subseteq \mathcal{I}$  be the parameters of  $\phi$ .

For each  $\sigma$  as above, we construct an indiscernible sequence  $\mathcal{J}_\sigma$  by induction...



## Stage 0

For all  $j < m + 1$ , choose  $\mathcal{I}_j^0$  to be a proper end segment of  $\mathcal{I}_j$  excluding  $B$  such that each  $\psi_\sigma$  is satisfied by every element of  $\mathcal{I}_{\sigma(0)}^0 \times \cdots \times \mathcal{I}_{\sigma(m-1)}^0$ .

Let  $\mathcal{I}^0 = \mathcal{I}$ , and let  $\mathcal{J}_\sigma^0 = \emptyset$  for each  $\sigma$ .



## Stage $2i + 1$

Let  $\mathcal{I}'$  be a finite subset of  $\mathcal{I}^{2i}$  containing  $B$ .

There is an increasing map

$$\mathcal{I}' \longrightarrow \mathcal{I} \setminus \bigcup_j \mathcal{I}_j^{2i}$$

fixing  $B$  such that for each  $j < m + 1$ , elements to the left of  $\mathcal{I}_j^{2i}$  remain to the left and all other elements map to the right of  $\mathcal{I}_j^{2i}$ .

This map extends to an automorphism fixing  $B$ , so by compactness, there is  $A' = (a'_0, \dots, a'_m)$  realizing  $\phi$  such that if we assign each  $a'_j$  to the cut of  $\mathcal{I}^{2i}$  immediately to the left of  $\mathcal{I}_j^{2i}$ , then any proper subsequence of  $A'$  inserts into  $\mathcal{I}^{2i} \subseteq \mathcal{I}$ .



## Stage $2i + 1$ (continued)

Recall

$$\phi(x_0, \dots, x_m) \vdash \bigvee_{\sigma} \neg \psi_{\sigma}(x_{\sigma(0)}, \dots, x_{\sigma(m-1)}). \quad (*)$$

We can choose  $\sigma_i : m \rightarrow m + 1$  increasing so that

$$a'_{\sigma_i(0)} \cdots a'_{\sigma_i(m-1)} \not\models \psi_{\sigma_i}.$$

Let

$$\mathcal{I}^{2i+1} = \mathcal{I}^{2i} \cup \{a'_{\sigma_i(j)} : j < m\}$$

where each  $a'_{\sigma_i(j)}$  is inserted immediately to the left of  $\mathcal{I}_{\sigma_i(j)}^{2i}$ . Let

$$\mathcal{J}_{\sigma_i}^{2i+1} = \mathcal{J}_{\sigma_i}^{2i} + (a'_{\sigma_i(0)}, \dots, a'_{\sigma_i(m-1)}).$$

For each  $j < m + 1$ , let  $\mathcal{I}_j^{2i+1} = \mathcal{I}_j^{2i}$ .



## Stage $2i + 2$

For each  $j < m + 1$ , choose  $b_j \in \mathcal{I}_j^{2i+1}$  and an end segment  $\mathcal{I}_j^{2i+2}$  of  $\mathcal{I}_j^{2i+1}$  excluding  $b_j$ .

Let  $\mathcal{I}^{2i+2} = \mathcal{I}^{2i+1}$ , and for each  $\sigma$ , let

$$\mathcal{J}_\sigma^{2i+2} = \mathcal{J}_\sigma^{2i+1} + \left( b_{\sigma(0)}, \dots, b_{\sigma(m-1)} \right).$$



## Proof of Lemma (continued)

For each  $\sigma$ , let  $\mathcal{J}_\sigma = \bigcup_{i < \omega} \mathcal{J}_\sigma^i$ .

Choose a  $\sigma$  which appears infinitely many times in  $(\sigma_i : i < \omega)$ .

It follows that  $\psi_\sigma$  alternates infinitely many times on  $\mathcal{J}_\sigma$ , contradicting NIP. ■

